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Covariance systems

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Abstract

We introduce new definitions of states and representations of covariance systems. The GNS-construction is generalized in this context. It associates a representation with each state of the covariance system. Next, the states are extended to the states of an appropriate covariance algebra. Two applications are given. We describe a non-relativistic quantum particle and give a simple description of the quantum spacetime model introduced by Doplicher *et al* (Doplicher *et al* 1994 *Phys. Lett.* B **331** 39, Doplicher *et al* 1995 *Commun. Math. Phys.* **172** 187).

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1. Introduction

Consider an action σ of a group X as automorphisms of a C^*-algebra $\mathcal{A}$. A $*$-representation π of $\mathcal{A}$ is X-*covariant* if a unitary representation U of X exists such that

$$\pi(\sigma_x a) = U(x)\pi(a)U(x)^* \tag{1}$$

holds for all $a \in \mathcal{A}$ and $x \in X$. A state ω of $\mathcal{A}$ is X-covariant if its GNS-representation is X-covariant. Doplicher *et al* [6] proved a one-to-one relation between covariant $*$-representations of $\mathcal{A}$ and $*$-representations of the crossed product algebra $\mathcal{A} \times_\sigma X$. In quantum mechanics, projective representations of the relevant symmetry group X are also important. A $*$-representation π of $\mathcal{A}$, which is only covariant in the sense that (1) holds w.r.t. a projective representation U of X, cannot extend to a $*$-representation of $\mathcal{A} \times_\sigma X$ because then U would not be projective. This leads to a situation where the covariance algebra $\mathcal{A} \times_\sigma X$ has to be replaced by some other algebra which depends in general on the choice of projective representation. An elegant formalism to deal with this situation is presented here. It introduces new concepts of representation and state of a covariance system. The GNS-construction, well known for states of C^*-algebras, is generalized and associates a representation with each state of the

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covariance system. Next, it is shown that each state of the covariance system extends to a state of an appropriate covariance algebra.

Our results on covariance systems combine well with a new C^*-algebraic approach to quantum mechanics and quantum field theory. It elaborates the usual approach as reviewed in [12]. The quantum mechanics of a single non-relativistic particle is described by the covariance system consisting of a group of spatial shifts $\mathbb{R}^3, +$, acting on an Abelian algebra of functions of position. This group can be extended with rotations to form the Euclidean group. If projective representations are allowed then the particle may have spin. Levy-Leblond [5] has shown, using the results of Bargmann [3], that the physically relevant representations of the Galilei group are projective representations, labelled by a free parameter, which is proportional to the mass of the particle. However, in non-relativistic quantum mechanics, time evolution is not a symmetry of the Abelian algebra of functions of position. For that reason, these projective representations of the Galilei group cannot be formulated as projective representations of a covariance system.

As a second example, we consider the model of quantum spacetime introduced by Doplicher *et al* [14, 15]. We restrict ourselves to the description of a single particle using the proper-time formalism of relativistic quantum mechanics. The configuration space Σ of its internal degrees of freedom is a four-dimensional manifold in $\mathbb{R}^6$. Shifts in Minkowski space and momentum space act on Σ in a trivial way. An explicit expression is given for a specific quasi-free state of the resulting covariance system. It involves a C^*-multiplier with values in the algebra of continuous complex functions of Σ. It has a non-trivial GNS-representation in which the shift groups have projective representations. Their generators are position and momentum operators satisfying non-canonical commutation relations. An alternative construction of a C^*-algebra with affiliated position operators satisfying the same non-canonical commutation relations is found in [16].

A third example, the electromagnetic radiation field, will be discussed elsewhere.

2. Covariance systems

2.1. C^*-multipliers

The following definition introduces a pair of objects (ξ, σ), where ξ generalizes the concept of multiplier to maps with values in a C^*-algebra, and where σ is a twisted representation of a group as automorphisms of a C^*-algebra. Operator-valued multipliers have been studied in [7, 18].

Definition 1. *A (left) C^*-multiplier of a locally compact group X with associated twisted representation σ is a measurable map ξ of $X \times X$ into the unitary elements of the multiplier algebra $M(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$ satisfying*

$$\xi(x, e) = \xi(e, y) = \mathbf{I} \qquad x, y \in X, \tag{2}$$

together with a map σ of X into the automorphisms of $\mathcal{A}$ such that σ_e is the identity transformation and

$$\sigma_x \xi(y, z) = \xi(x, y)\xi(xy, z)\xi(x, yz)^* \qquad x, y, z \in X \tag{3}$$

and

$$\sigma_x \sigma_y a = \xi(x, y)(\sigma_{xy} a)\xi(x, y)^* \qquad x, y \in X \quad a \in \mathcal{A}. \tag{4}$$

If the C^*-algebra $\mathcal{A}$ has no unit, it has *a fortiori* no unitary elements. For this reason we make use of the multiplier algebra $M(\mathcal{A})$ of $\mathcal{A}$—see e.g. [17, section IV.2]. It is isomorphic to

$\mathcal{A}$ if the latter has a unit. It is in any case isomorphic to the algebra of bounded operators B for which $B\pi(\mathcal{A}) \subset \pi(\mathcal{A})$, where π is a faithful non-degenerate representation of $\mathcal{A}$ as bounded operators of a Hilbert space.

Throughout this paper we assume that ξ is continuous in a neighbourhood of the neutral element e of X. If $\xi(x, y)$ is a multiple of $\mathbf{I}$ for all $x, y \in X$ then ξ is also called a *cocycle*.

A right C^*-multiplier ζ satisfies

$$\sigma_y^{-1}\zeta(z, x) = \zeta(zx, y)^*\zeta(z, xy)\zeta(x, y) \qquad x, y, z \in X \tag{5}$$

instead of (3) and

$$\sigma_x\sigma_y a = \sigma_{xy}(\zeta(x, y)a\zeta(x, y)^*) \qquad x, y \in X, a \in \mathcal{A} \tag{6}$$

instead of (4).

In what follows we need only the multipliers for which the associated representation σ is an action of the group X as automorphisms of $\mathcal{A}$. In that case, (4) means that $\xi(x, y)$ belongs to the centre of $\mathcal{M}(\mathcal{A})$.

2.2. Covariant states

The following definition can be found in the literature, see e.g. [10].

Definition 2. *A covariance system is a triple $(\mathcal{A}, X, \sigma)$ consisting of a C^*-algebra $\mathcal{A}$, a locally compact symmetry group X and a strongly continuous action σ of X as automorphisms of $\mathcal{A}$.*

Strong continuity of σ means that the map $x \in X \to \sigma_x a$ is continuous in norm for each a in $\mathcal{A}$. The definition generalizes that of a dynamical system as often found in the literature (take $X = \mathbb{R}$, +, and $t \in \mathbb{R} \to \sigma_t$ is the time evolution of the quantum system). Here we have rather different applications in mind. Examples are given further on. Note that some authors [11] use the name C^*-dynamical system with the same meaning as covariance system.

The following definitions are new. For convenience we introduce

Definition 3. *A multiplier ξ of a covariance system $(\mathcal{A}, X, \sigma)$ is a C^*-multiplier of X, with values in the centre of $\mathcal{M}(\mathcal{A})$, whose associated twisted representation coincides with σ.*

The reason for restricting the values of ξ to the centre of $\mathcal{M}(\mathcal{A})$ is that the representation σ of X is not twisted. With a left multiplier ξ of $(\mathcal{A}, X, \sigma)$ corresponds a right multiplier ζ defined by

$$\zeta(x, y) = \sigma_{xy}^{-1}\xi(x, y). \tag{7}$$

Throughout this paper the symbols ξ and ζ are related in this way.

The notion of a state of a C^*-algebra is well known. It is generalized here as follows.

Definition 4. *A state of the covariance system $(\mathcal{A}, X, \sigma)$ is a measurable map ω of $X \times X$ into continuous linear complex-valued functions of $\mathcal{A}$ having the following properties.*

- *(positivity) For all $n > 0$ and for all possible choices of $\lambda_1, \ldots, \lambda_n$ in $\mathbb{C}$, of $x_1, \ldots, x_n$ in X and of $a_1, \ldots a_n$ in $\mathcal{A}$ is*

$$\sum_{j,k=1}^{n} \lambda_j\bar{\lambda}_k\omega_{x_j,x_k}(a_k^*a_j) \geqslant 0. \tag{8}$$

- *(normalization) $\omega_{e,e}$ is a state of $\mathcal{A}$.*
- *(covariance) There exists a right multiplier ζ of $(\mathcal{A}, X, \sigma)$ satisfying*

$$\omega_{x,y}(\sigma_z a) = \omega_{xz,yz}(\zeta(y, z)a\zeta(x, z)^*) \tag{9}$$

for all $x, y, z \in X$ and $a \in \mathcal{A}$.

- *(continuity) For any $a \in \mathcal{A}$ the map $x, y \to \omega_{x,y}(a)$ is continuous in a neighbourhood of the neutral element of $X \times X$.*

The state ω is faithful *if $\omega_{e,e}$ is faithful.*

An alternative notation is

$$F(a, x, y) = \omega_{x,y}(a). \tag{10}$$

In the case that the C^*-algebra $\mathcal{A}$ is trivial (i.e. equal to $\mathbb{C}$), then the first argument may be omitted. The resulting function $F(x, y)$ is the generalization of Wigner's function, or its Fourier transform, the characteristic function of Moyal [2].

Any X-covariant state ω of the C^*-algebra $\mathcal{A}$ defines a state (again denoted by ω) of the covariance system $(\mathcal{A}, X, \sigma)$ by

$$\omega_{x,y}(a) = (\pi(a)U(x)^*\psi, U(y)^*\psi) \qquad x, y \in X \quad a \in \mathcal{A} \tag{11}$$

where $(\mathcal{H}, \pi, \psi)$ is the GNS-representation of $\mathcal{A}$ induced by ω and U is the covariant representation of X. The converse is also true. If ω is a state of $(\mathcal{A}, X, \sigma)$ then $\omega_{e,e}$ is an X-covariant state of $\mathcal{A}$ provided we weaken the definition of X-covariance to allow for projective representations. This will be shown in section 2.5.

The following results are needed for technical reasons. The next lemma generalizes the lemma of Schwarz.

Lemma 1. *Let ω be a state of $(\mathcal{A}, X, \sigma)$. Then*

$$\left|\sum_{j,k=1}^{n} \lambda_j \bar{\lambda}_k \omega_{x_j,x_k}(a_k^* b_j - b_k^* a_j)\right|^2 \leqslant 4 \sum_{j,k=1}^{n} \lambda_j \bar{\lambda}_k \omega_{x_j,x_k}(a_k^* a_j) \sum_{j,k=1}^{n} \lambda_j \bar{\lambda}_k \omega_{x_j,x_k}(b_k^* b_j) \tag{12}$$

for all choices of $\lambda_1, \ldots, \lambda_n$ in $\mathbb{C}$, of $x_1, \ldots, x_n$ in X and of $a_1, \ldots a_n$ and $b_1, \ldots, b_n$ in $\mathcal{A}$.

The proof is straightforward. The lemma is now used to prove uniform continuity of $\omega_{x,y}$.

Proposition 1. *Let ω be a state of $(\mathcal{A}, X, \sigma)$. One has*

$$\sum_{j,k=1}^{n} \lambda_j \bar{\lambda}_k \omega_{x_j,x_k}(a_k^* a_j) \leqslant \left(\sum_{j=1}^{n} |\lambda_j| \, \|a_j\|\right)^2 \tag{13}$$

for all choices of $\lambda_1, \ldots, \lambda_n$ in $\mathbb{C}$, of $x_1, \ldots, x_n$ in X and of $a_1, \ldots a_n$ in $\mathcal{A}$.

Proof. The statement is clearly true for $n = 1$. Assume it to hold up to $n - 1$. One calculates, using the lemma with $b_k = \mathrm{i}\delta_{k,n} a_n$,

$$\begin{aligned}\sum_{j,k=1}^{n} \lambda_j \bar{\lambda}_k \omega_{x_j,x_k}(a_k^* a_j) &\leqslant \left(\sum_{j=1}^{n-1} |\lambda_j| \, \|a_j\|\right)^2 + \sum_{j=1}^{n-1} \lambda_j \bar{\lambda}_n \omega_{x_j,x_n}(a_n^* a_j) \\ &\quad + \sum_{j=1}^{n-1} \lambda_n \bar{\lambda}_j \omega_{x_n,x_j}(a_j^* a_n) + |\lambda_n|^2 \, \|a_n\|^2 \\ &\leqslant \left(\sum_{j=1}^{n} |\lambda_j| \, \|a_j\|\right)^2.\end{aligned} \tag{14}$$

Hence the proof follows by induction. □

2.3. *Representations of a covariance system*

The following two definitions are obvious.

Definition 5. *A representation of the covariance system* $(\mathcal{A}, X, \sigma)$ *is a triple* $(\mathcal{H}, \pi, U)$ *which consists of a* $*$*-representation* π *of* $\mathcal{A}$ *in a Hilbert space* $\mathcal{H}$ *and a measurable map* $x \in X \to U(x)$ *into the unitary operators of* $\mathcal{H}$*, with the properties that (1) holds, and that each normalized element* ψ *of* $\mathcal{H}$ *defines a state* ω *of the covariance system by (11). An element* $\psi \in \mathcal{H}$ *is cyclic for the representation if the subspace spanned by*

$$\{\pi(a)U(x)^*\psi \,:\, a \in \mathcal{A}, x \in X\} \tag{15}$$

is dense in $\mathcal{H}$.

Definition 6. *Two representations* $(\mathcal{H}, \pi, U)$ *and* $(\mathcal{H}', \pi', U')$ *of* $(\mathcal{A}, X, \sigma)$ *are equivalent if there exists an isomorphism* V *of* $\mathcal{H}$ *onto* $\mathcal{H}'$ *intertwining* π *and* π'*, respectively* U *and* U'.

2.4. *Projective representations of X*

Proposition 2. *Consider a multiplier* ξ *of a covariance system* $(\mathcal{A}, X, \sigma)$*. Let* π *be a* $*$*-representation of* $\mathcal{A}$ *in a Hilbert space* $\mathcal{H}$ *and let* $x \to U(x)$ *be a projective representation of* X *in* $\mathcal{H}$ *in the sense that (1)* $U(e) = \mathbf{I}$*; (2)* $U(x)$ *is unitary for all* $x \in X$*; (3) the composition law is given by*

$$U(x)U(y) = \pi(\xi(x, y))U(xy) \qquad x, y \in X. \tag{16}$$

Assume further that $x \to U(x)$ *is strongly continuous in a neighbourhood of the neutral element of* X *and that the covariance condition (1) is satisfied. Then* $(\mathcal{H}, \pi, U)$ *is a representation of the covariance system* $(\mathcal{A}, X, \sigma)$.

Proof. We have to show that each normalized element ψ of $\mathcal{H}$ defines a state ω of the covariance system by (11). Positivity, normalization and continuity are straightforward. The covariance condition remains to be shown. Note that

$$U(xy)^*U(x)U(y) = \pi(\zeta(x, y)) \tag{17}$$

with ζ related to ξ by (7). Using this result covariance follows from

$$\begin{aligned}\omega_{x,y}(\sigma_z a) &= (\pi(\sigma_z a)U(x)^*\psi, U(y)^*\psi)\\ &= (\pi(a)U(z)^*U(x)^*\psi, U(z)^*U(y)^*\psi)\\ &= (\pi(a)(U(xz)\pi(\zeta(x, z)))^*\psi, (U(yz)\pi(\zeta(y, z)))^*\psi)\\ &= \omega_{xz,yz}(\zeta(y, z)a\zeta(x, z)^*).\end{aligned} \tag{18}$$

□

2.5. *GNS-construction*

The GNS-construction for the states of a C^*-algebra can be generalized as follows.

Theorem 1. *Let* ω *be a state of the covariance system* $(\mathcal{A}, X, \sigma)$*. There exists a representation* $(\mathcal{H}, \pi, U)$ *of* $(\mathcal{A}, X, \sigma)$ *and a multiplier* ξ *of* $(\mathcal{A}, X, \sigma)$ *such that*

- *U is a projective representation of X satisying (16),*
- *(9) holds with* ζ *related to* ξ *by (7),*
- *There exists a cyclic vector* ψ *of* $\mathcal{H}$ *such that (11) holds.*

The quadruple $(\mathcal{H}, \pi, U, \psi)$ *is unique up to equivalence of representations, i.e. if* $(\mathcal{H}', \pi', U')$ *is a representation and* ψ' *is a cyclic vector of* $\mathcal{H}'$ *satisfying (11) with* ψ *replaced by* ψ' *and* π *and* U *replaced by* π' *and* U'*, then there exists an isomorphism* V *of* $\mathcal{H}$ *onto* $\mathcal{H}'$ *intertwining* π *and* π'*, respectively* U *and* U'*, and mapping* ψ *onto* ψ'*.*

Proof. Let $\mathcal{C}_c(X, \mathcal{A})$ denote the linear space of the continuous functions with compact support in X and with the values in $\mathcal{A}$. A sesquilinear form is defined by

$$(f, g) = \int_X \mathrm{d}x\,\Delta(x)^{-1} \int_X \mathrm{d}y\,\Delta(y)^{-1}\, \omega_{x,y}(g(y)^* f(x)) \tag{19}$$

for all $f,\ g \in \mathcal{C}_c(X, \mathcal{A})$ (Δ is the modular function of X). From the positivity of ω it follows that $(\cdot, \cdot)$ is a positive form. Let us assume for simplicity of notation that $(\cdot, \cdot)$ is not degenerate (it is easy to see that further definitions do not depend on the choice of the representative of the equivalent class given by the kernel of $(\cdot, \cdot)$). Then it is an inner product making $\mathcal{C}_c(X, \mathcal{A})$ into a pre-Hilbert space. Let $\mathcal{H}$ denote its completion.

Define π by

$$\pi(a) f(x) = a f(x) \qquad a \in \mathcal{A} \quad f \in \mathcal{C}_c(X, \mathcal{A}) \quad x \in X. \tag{20}$$

By linearity $\pi(a)$ extends to a linear operator with the domain $\mathcal{C}_c(X, \mathcal{A})$. It is obvious that $\pi(a)\pi(b) = \pi(ab)$, π is linear and $\pi(b)^* = \pi(b^*)$. If $a \geqslant 0$ then

$$(\pi(a) f, f) = \int_X \mathrm{d}x\,\Delta(x)^{-1} \int_X \mathrm{d}y\,\Delta(y)^{-1}\, \omega_{x,y}(f(y)^* a f(x)) \geqslant 0. \tag{21}$$

This implies that $\pi(a)$ is bounded for all $a \in \mathcal{A}$ (indeed, (20) extends to the multiplier algebra $\mathcal{M}(\mathcal{A})$ and $\pi(\mathbf{I}) = \mathbf{I}$ so that $\|a\|^2 - a^* a \geqslant 0$ implies that $\|a\|^2\, \|f\|^2 - \|\pi(a) f\|^2 \geqslant 0$). Hence π is a *-representation of $\mathcal{A}$ in $\mathcal{H}$.

Next define a linear operator U by

$$U(x) f(y) = \sigma_x\,[f(yx)\zeta(y, x)] \qquad x, y \in X \quad f \in \mathcal{C}_c(X, \mathcal{A}). \tag{22}$$

Note that one has $U(e) = \mathbf{I}$. A straightforward calculation gives

$$U(x)^* f(y) = (\sigma_x^{-1} f(yx^{-1}))\zeta(yx^{-1}, x)^*. \tag{23}$$

This expression can be used to verify that $U(x)$ is unitary.

Let ξ be defined by (7). Then a short calculation using (5) gives

$$\begin{aligned} U(x)U(y) f(z) &= \sigma_x \left[\sigma_y\,[f(zxy)\zeta(zx, y)]\,\zeta(z, x)\right] \\ &= \sigma_{xy}\,[f(zxy)\zeta(x, y)\zeta(z, xy)] \\ &= \pi(\xi(x, y))U(xy) f(z) \end{aligned} \tag{24}$$

which is (16).

Let $(u_\alpha)_\alpha$ be an approximate unit of $\mathcal{A}$. For each neighbourhood v of e in X, let δ_v be a positive function vanishing outside v and satisfying

$$\int_X \mathrm{d}x\,\delta_v(x) = 1. \tag{25}$$

Then the functions $(\delta_v u_\alpha)_{v,\alpha}$ form a Cauchy net in $\mathcal{H}$ converging to some element ψ.

From (23), it now follows that

$$\int_X \mathrm{d}x\,\Delta(x)^{-1}\pi(f(x))U(x)^*\psi = f \tag{26}$$

for all $f \in \mathcal{C}_c(X, \mathcal{A})$. This proves that ψ is cyclic. The latter also implies that

$$(f, g) = \int_X \mathrm{d}x\,\Delta(x)^{-1} \int_X \mathrm{d}y\,\Delta(y)^{-1}(\pi(g(y)^* f(x))U(x)^*\psi, U(y)^*\psi). \tag{27}$$

Comparison with (19) shows that (11) holds. From (22) and (23) it follows immediately that (1) holds. Hence, proposition 2 asserts that $(\mathcal{H}, \pi, U)$ is a representation of the covariance system.

Finally, we prove uniqueness up to equivalence of representations. Define V with domain $\mathcal{C}_c(X, \mathcal{A})$ by

$$Vf = \int_X \mathrm{d}x\, \Delta(x)^{-1}\pi'(f(x))U'(x)^*\psi'. \tag{28}$$

It is straightforward to verify that V extends to an isometry of $\mathcal{H}$ into $\mathcal{H}'$. Because ψ' is cyclic, V is an isomorphism.

That V intertwines π and π' is obvious. For U and U' one has

$$\begin{aligned} U'(x)Vf &= U'(x)\int_X \mathrm{d}y\, \Delta(y)^{-1}\pi'(f(y))U'(y)^*\psi' \\ &= U'(x)\int_X \mathrm{d}y\, \Delta(y)^{-1}\pi'(f(yx))U'(yx)^*\psi' \\ &= U'(x)\int_X \mathrm{d}y\, \Delta(y)^{-1}\pi'(f(yx)\zeta(y,x))U'(x)^*U'(y)^*\psi' \\ &= \int_X \mathrm{d}y\, \Delta(y)^{-1}\pi'\left(\sigma_x\left[f(yx)\zeta(y,x)\right]\right)U'(y)^*\psi' \\ &= VU(x)f. \end{aligned} \tag{29}$$

Finally, it is obvious that $V\psi = \psi'$. □

2.6. Crossed product algebras

A reader not interested in crossed product algebras may skip this section. It is not needed for the sequel of this paper but is added to clarify the relation between the present and previous work on crossed product algebras. Details found in [18] are not repeated here.

Let ω be a given state of a covariance system $(\mathcal{A}, X, \sigma)$. Let ζ be the right multiplier associated with ω. Recall that σ is the representation associated with ζ. Let ξ be the left multiplier derived from ζ by (7). The representation σ together with ξ determines a crossed product algebra $\mathcal{A} \times_\xi X$. It is constructed as follows. Let $\mathcal{L}_1(X, \mathcal{A})$ denote the linear space of integrable functions of X with values in $\mathcal{A}$. A product law for the elements of $\mathcal{L}_1(X, \mathcal{A})$ is given by

$$(f \times g)(x) = \int_X \mathrm{d}y\, f(y)\xi(y, y^{-1}x)\sigma_y g(y^{-1}x). \tag{30}$$

An involution is given by

$$f^\star(x) = \Delta(x)^{-1}\xi(x, x^{-1})^*\sigma_x f(x^{-1})^*. \tag{31}$$

In this way $\mathcal{L}_1(X, \mathcal{A})$ becomes an involutive algebra. By closure in an appropriate norm it becomes the C^*-algebra $\mathcal{A} \times_\xi X$.

A linear functional $\bar{\omega}$ of $\mathcal{L}_1(X, \mathcal{A})$ is defined by

$$\bar{\omega}(f) = \int_X \mathrm{d}x\, \Delta(x)^{-1}\omega_{x,e}(\xi(x^{-1}, x)f(x^{-1})). \tag{32}$$

Proposition 3. *$\bar{\omega}$ extends to a state of $\mathcal{A} \times_\xi X$.*

Proof. Positivity is verified as follows:

$$\begin{aligned}
\bar{\omega}(f^\star \times f) &= \int_X \mathrm{d}x\, \Delta(x)^{-1}\omega_{x,e}(\xi(x^{-1},x)(f^\star \times f)(x^{-1}))\\
&= \int_X \mathrm{d}x\, \Delta(x)^{-1}\int_X \mathrm{d}y\, \Delta(y)^{-1}\\
&\quad\times \omega_{x,e}(\xi(x^{-1},x)\xi(y,y^{-1})^*(\sigma_y f(y^{-1}))^*\xi(y,y^{-1}x^{-1})\sigma_y f(y^{-1}x^{-1}))\\
&= \int_X \mathrm{d}x\, \Delta(x)^{-1}\int_X \mathrm{d}y\, \Delta(y)^{-1}\\
&\quad\times \omega_{x,e}(\sigma_y[\xi(y^{-1}x^{-1},x)f(y^{-1}))^* f(y^{-1}x^{-1})])\\
&= \int_X \mathrm{d}x\, \Delta(x)^{-1}\int_X \mathrm{d}y\, \Delta(y)^{-1}\\
&\quad\times \omega_{xy,y}(\xi(y^{-1}x^{-1},x)\zeta(x,y)^* f(y^{-1}))^* f(y^{-1}x^{-1}))\\
&= \int_X \mathrm{d}x\, \Delta(x)^{-1}\int_X \mathrm{d}y\, \Delta(y)^{-1}\\
&\quad\times \omega_{xy,y}(\xi(y^{-1},y)\xi(y^{-1}x^{-1},xy)^* f(y^{-1}))^* f(y^{-1}x^{-1}))\\
&= \int_X \mathrm{d}x\, \Delta(x)^{-1}\int_X \mathrm{d}y\, \Delta(y)^{-1}\\
&\quad\times \omega_{x,y}(\xi(y^{-1},y)\xi(x^{-1},x)^* f(y^{-1}))^* f(x^{-1})) \geqslant 0. \qquad (33)
\end{aligned}$$

For each neighbourhood v of the neutral element of X, let δ_v be a positive normalized function with support in v. Then $(\delta_v u_\alpha)_{v,\alpha}$ is an approximate unit of $\mathcal{L}_1(X,\mathcal{A})$. One has

$$\bar{\omega}(\delta_v u_\alpha) = \int_X \mathrm{d}x\, \Delta(x)^{-1}\delta_v(x^{-1})\omega_{x,e}(\xi(x^{-1},x)u_\alpha). \qquad (34)$$

The latter tends to 1 because of the continuity of $x \to \omega_{x,e}$ and $x \to \xi(x^{-1},x)$ in the vicinity of e.

One concludes that $\bar{\omega}$ is a positive normalized linear functional of the involutive algebra $\mathcal{L}_1(X,\mathcal{A})$. By continuity it extends to a state of $\mathcal{A} \times_\xi X$. □

Now let $(\mathcal{H},\pi,U,\psi)$ be the GNS-representation of $(\mathcal{A},X,\sigma)$ induced by ω. Then one has

$$\begin{aligned}
\bar{\omega}(f) &= \int_X \mathrm{d}x\, \Delta(x)^{-1}\omega_{x,e}(\xi(x^{-1},x)f(x^{-1}))\\
&= \int_X \mathrm{d}x\, \Delta(x)^{-1}(\pi(\xi(x^{-1},x))\pi(f(x^{-1}))U(x)^*\psi,\psi)\\
&= \int_X \mathrm{d}x\, (\pi(f(x))U(x)\psi,\psi). \qquad (35)
\end{aligned}$$

A representation $\bar{\pi}$ of $\mathcal{A} \times_\xi X$ is now defined by the extension of

$$\bar{\pi}(f) = \int_X \mathrm{d}x\, \pi(f(x))U(x) \qquad f \in \mathcal{L}_1(X,\mathcal{A}). \qquad (36)$$

It is obvious that this representation is equivalent to the GNS-representation of $\bar{\omega}$. Conversely, if $\bar{\omega}$ is any state of $\mathcal{A} \times_\xi X$ then a state ω of the covariance system $(\mathcal{A},X,\sigma)$ can be defined by (9) using the GNS-representation of $\bar{\omega}$ (see the remarks after theorem 2 of [18]). This representation is then equivalent to the GNS-representation of ω.

3. Non-relativistic particle

This section is devoted to the projective representations occurring in the quantum mechanics of a non-relativistic particle and serves as an example of our approach. It relies partly on older work by Levy-Leblond [5], work which has been extended by Hagen [8, 9].

The algebra $\mathcal{A}$ is the algebra $\mathcal{C}_0(\mathbb{R}^n)$ of classical functions of position. The appropriate group of symmetry transformations is the Euclidean group. However, it is easier to start with the subgroup $\mathbb{R}^n$, + of shifts. This already yields the standard representation of quantum mechanics. Later on, rotations are added to discuss the spin of the particle. Finally, transformations to a moving frame are discussed.

3.1. Standard representation

First assume $X = \mathbb{R}^n, +$. The shifts act as automorphisms of $\mathcal{A}$ by

$$(\sigma_q f)(q') = f(q' - q) \qquad f \in \mathcal{A} \quad q, q' \in X. \tag{37}$$

A $*$-representation π of $\mathcal{A}$ as bounded operators of $\mathcal{H} = \mathcal{L}_2(\mathbb{R}^n)$ is defined by

$$\pi(f)\psi(q) = f(q)\psi(q) \qquad f \in \mathcal{A} \quad q \in \mathbb{R}^n. \tag{38}$$

A unitary representation U of X is defined by

$$U(q)\psi(q') = \psi(q' - q) \qquad q, q' \in \mathbb{R}^n. \tag{39}$$

One immediately verifies that $(\mathcal{H}, \pi, U)$ is a representation of $(\mathcal{A}, X, \sigma)$. In particular, any normalized element ψ of $\mathcal{H}$ defines a state of $(\mathcal{A}, X, \sigma)$. The multiplier ξ associated with such a state is identically equal to 1.

Using Stone's theorem, the shift operators can be written as

$$U(q) = \exp\left(-(\mathrm{i}/\hbar)\sum_{j=1}^{n} q_j P_j\right) \tag{40}$$

with P_j the momentum operators and $\hbar$ equal to Planck's constant divided by 2π. A short calculation gives the canonical commutation relations

$$\left[Q_j, P_k\right]_- = \delta_{j,k}\mathrm{i}\hbar \tag{41}$$

with Q_J the multiplication operators

$$Q_j\psi(q) = q_j\psi(q) \tag{42}$$

defined on a suitable domain.

3.2. Rotation symmetry

Now we extend the symmetry group X to the Euclidean group by adding rotations. We first discuss a particle without spin.

The group law is

$$(q', \Lambda')(q, \Lambda) = (q' + \Lambda' q, \Lambda'\Lambda) \qquad q, q' \in \mathbb{R}^n \quad \Lambda, \Lambda' \in SO(n). \tag{43}$$

The representation σ is given by

$$(\sigma_{q,\Lambda} f)(q') = f(\Lambda^{-1}(q' - q)) \qquad f \in \mathcal{A} \quad q, q' \in \mathbb{R}^n \quad \Lambda \in SO(n). \tag{44}$$

The $*$-representation $(\mathcal{H}, \pi)$ is the standard representation of the previous section. The unitary representation U of X is given by

$$(U(q, \Lambda)\psi)(q') = \psi(\Lambda^{-1}(q' - q)) \tag{45}$$

with $\psi \in \mathcal{L}_2(\mathbb{R}^n)$ and $q, q' \in \mathbb{R}^n, \Lambda \in SO(n)$. Then $(\mathcal{H}, \pi, U)$ is a representation of $(\mathcal{A}, X, \sigma)$. In particular, any normalized element ψ of $\mathcal{H}$ defines a state of $(\mathcal{A}, X, \sigma)$. The multiplier ξ associated with such a state is identically equal to 1.

3.3. Spin

Fix $n = 3$. The covariance system $(\mathcal{A}, X, \sigma)$ of the previous section also has states whose associated multiplier ξ is non-trivial. The representation induced by these states is the so-called spinor representation. The reason for their existence is that the group $SO(3)$ is doubly connected with covering group $SU(2)$.

Given $q \in \mathbb{R}^3$, construct the matrix

$$M(q) = \sum_{j=1}^{3} q_j \sigma_j \tag{46}$$

with $\sigma_1, \sigma_2, \sigma_3$ the three Pauli matrices. The matrix $M(q)$ transforms under an element u of $SU(2)$ into the matrix $M(q') = uM(q)u^*$. It is easy to show that the transformation $q \to q'$ is a rotation, i.e. there exists $\Xi(u) \in SO(3)$ for which $q' = \Xi(u)q$. Note that Ξ is a homomorphism of $SU(2)$ onto $SO(3)$. Let $\Lambda \to v(\Lambda)$ be an inverse of $u \to \Xi(u)$ in the sense that $\Xi(v(\Lambda)) = \Lambda$ for all Λ and $v(\Xi(u)) = u$ for all u in the neighbourhood of the identity matrix. Note that the map v cannot be continuous. A cocycle ξ of $SO(3)$ is defined by

$$v(\Lambda)v(\Lambda') = \xi(\Lambda, \Lambda')v(\Lambda\Lambda'). \tag{47}$$

From

$$\Xi(v(\Lambda)v(\Lambda')) = \Lambda\Lambda' \tag{48}$$

and

$$\Xi(\xi(\Lambda, \Lambda')v(\Lambda\Lambda')) = \Xi(\xi(\Lambda, \Lambda'))\Lambda\Lambda' \tag{49}$$

it follows that $\Xi(\xi(\Lambda, \Lambda')) = \mathbf{I}$. This implies that $\xi(\Lambda, \Lambda') = \pm 1$. Hence it is clear that ξ is a multiplier of $(\mathcal{A}, X, \sigma)$.

Consider the Hilbert space $\mathcal{H} = \mathcal{L}_2(\mathbb{R}^n) \oplus \mathcal{L}_2(\mathbb{R}^n)$. An element of this space is denoted by $\binom{\psi_1}{\psi_2}$. It is normalized if $\|\psi_1\|^2 + \|\psi_2\|^2 = 1$. A $*$-representation π of $\mathcal{A}$ is defined by

$$\pi(f)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}(q) = f(q)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}(q). \tag{50}$$

A projective representation U of X is defined by

$$U(q, \Lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}(q') = v(\Lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}(\Lambda^{-1}(q' - q)). \tag{51}$$

Indeed, one verifies that

$$\begin{aligned} U(q', \Lambda')U(q, \Lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}(q'') &= v(\Lambda')v(\Lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}(\Lambda^{-1}(\Lambda'^{-1}(q'' - q') - q)) \\ &= \xi(\Lambda', \Lambda)v(\Lambda'\Lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}((\Lambda'\Lambda)^{-1}(q'' - q' - \Lambda' q) \\ &= \xi(\Lambda', \Lambda)U(q' + \Lambda' q, \Lambda'\Lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}(q''). \end{aligned} \tag{52}$$

Because (1) is satisfied it follows from property 2 that $(\mathcal{H}, \pi, U)$ is a representation of $(\mathcal{A}, X, \sigma)$.

Each normalized element $\binom{\psi_1}{\psi_2}$ of $\mathcal{H}$ determines a state ω of $(\mathcal{A}, X, \sigma)$. In what follows we use the F-notation (10). A tedious calculation yields

$$F(f; q, \Lambda; q', \Lambda') = \int_{\mathbb{R}^3} \mathrm{d}q'' \, f(q'') \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (\Lambda' q'' + q') \cdot v(\Lambda') v(\Lambda)^* \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (\Lambda q'' + q). \tag{53}$$

In particular, there follows

$$F(f; q, \Lambda; q, \Lambda) = \int_{\mathbb{R}^3} \mathrm{d}q'' \, f(q'')(|\psi_1|^2 + |\psi_2|^2)(\Lambda q'' + q). \tag{54}$$

These diagonal elements of ω do not reveal whether the state has spin or not. Only the correlation effects visible in the off-diagonal elements can reveal the presence of mechanical spin. Consider, for example, a rotation by π around the z-axis (denoted by Λ). Then one has $v(\Lambda) = \sigma_z$ so that

$$F(f; q, \Lambda; 0, \mathbb{I}) = \int_{\mathbb{R}^3} \mathrm{d}q'' \, f(q'')(\bar{\psi}_1(q'')\psi_1(\Lambda q'' + q) - \bar{\psi}_2(q'')\psi_2(\Lambda q'' + q)). \tag{55}$$

The minus sign is a consequence of spin and has the effect that the particle observed in a frame rotated by 180° does not look as expected.

3.4. Mass

Consider now the full Galilei group G. Its composition law is

$$(q', \Lambda', t', v')(q, \Lambda, t, v) = (q' + \Lambda' q + t v', \Lambda' \Lambda, t' + t, v' + \Lambda' v). \tag{56}$$

As before $q \in \mathbb{R}^n, +$ is a shift operation and $\Lambda \in SO(3)$ is a rotation. The remaining parameters describe a transformation to a moving frame with velocity $v \in \mathbb{R}^n$ and time $t \in \mathbb{R}$.

For simplicity we consider a particle without spin. Then the relevant $*$-representation π of $\mathcal{A}$ is the standard representation of section 3.1. The Galilei group has non-trivial cocycles ξ_κ parametrized by some parameter $\kappa \neq 0$ [3, 5]. A possible definition of ξ_κ is

$$\xi_\kappa(q', \Lambda', t', v'; q, \Lambda, t, v) = \exp\left(-\mathrm{i}\kappa \left[\frac{1}{2} t' |v|^2 + t' v' \cdot \Lambda' v - v' \cdot \Lambda'(q - t v)\right]\right). \tag{57}$$

It appears in the following projective representation of G

$$U(q, \Lambda, t, v) = U(q - t v, \Lambda) \, \exp(\mathrm{i}\kappa v \cdot \Lambda Q) \, \exp(\mathrm{i}\hbar^{-1} t H) \tag{58}$$

where H is the Hamiltonian of the free particle

$$H = \frac{1}{2m} P^2. \tag{59}$$

The mass m is related to κ by $m = \hbar\kappa$, P and Q are the position and momentum operators (see section 3.1) and $U(q, \Lambda)$ is the unitary representation defined by (45). Indeed, a tedious calculation shows that U, defined in this way, satisfies (16) with $\xi = \xi_\kappa$.

The projective representation (58) does not determine a projective representation of a covariance system for the simple reason that time evolution is not a symmetry of the algebra $\mathcal{A}$ of functions of position. For this purpose $\mathcal{A}$ should be extended to an algebra of functions of position and time. The natural context for doing so is relativistic quantum mechanics. In the present non-relativistic case, the representation (58) determines in an obvious way a projective representation of the enlarged covariance system.

4. Non-canonical commutation relations

In this section, we construct a variant of the model introduced by Doplicher *et al* [14, 15]. It describes a single-quantum particle in spacetime using the off-mass-shell formalism. A preliminary version of our work on this model has been reported in [18].

We do not follow the usual notational conventions of relativity theory. In particular, the inner product $x \cdot y$ is that of $\mathbb{R}^4$ and not that of Minkowski space. The reason for doing so is that we use simultaneously two different metric tensors, denoted respectively by g and γ.

4.1. Model

Let us make the (quite unusual) *ansatz* that a relativistic particle is characterized by two elements e and m of $\mathbb{R}^3$ satisfying $|e|^2 = |m|^2$ and $e \cdot m = \pm 1$. Let $\Sigma \subset \mathbb{R}^6$ denote the space of these pairs (e, m). It is, by assumption, the classical configuration space of the particle. See [14, 15] for a motivation of this particular choice. Note that Σ is a locally compact manifold in $\mathbb{R}^6$. The algebra $\mathcal{A}$ of the classical observables equals the C^*-algebra $\mathcal{C}_0(\Sigma)$ of continuous complex functions of Σ vanishing at infinity. Consider as a symmetry group X of Σ the group $\mathbb{R}^4 \times \mathbb{R}^4$, + acting in a trivial way, i.e. $\sigma_{k,q} f = f$ for all $k, q \in \mathbb{R}^4$ and $f \in \mathcal{C}_0(\Sigma)$. In this way one obtains a covariance system $(\mathcal{A}, X, \sigma)$.

The metric tensor of Minkowski space is denoted by g. It is the diagonal matrix [1, −1, −1, −1]. The geometry of spacetime is described by a metric tensor $\gamma(e, m)$ which is an invertible 4×4 matrix with real coefficients depending continuously on (e, m). A possible choice is $\gamma = g$.

For each $(e, m) \in \Sigma$, form an antisymmetric matrix $\epsilon(e, m)$ by

$$\epsilon(e, m) = \begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & m_3 & -m_2 \\ -e_2 & -m_3 & 0 & m_1 \\ -e_3 & m_2 & -m_1 & 0 \end{pmatrix}. \tag{60}$$

The matrix $\epsilon^{-1}\gamma$ will appear often. It transforms the positions q into the wave vectors k. Let us introduce the notation

$$\eta(e, m) = (e \cdot m)\epsilon^{-1}(e, m)\gamma(e, m). \tag{61}$$

The factor $(e \cdot m)$, which equals ± 1, has been included for physical reasons (one expects that the wave vector k will transform into gk under time reversal, while q transforms into $-gq$).

Later on, the group X will be extended to the Lorentz group. The action of the Lorentz group on the functions of Σ will be non-trivial. Since integration over Σ should be compatible with this action, we introduce it here. A representation σ of the Lorentz group as automorphisms of $\mathcal{C}_0(\Sigma)$ is defined by

$$\sigma_\Lambda f(e, m) = f(e', m') \qquad \text{with} \qquad \epsilon(e', m') = \tilde{\Lambda}\epsilon(e, m)\Lambda. \tag{62}$$

Here, $\tilde{\Lambda}$ denotes the transpose of the matrix Λ. An invariant measure ρ_0 of Σ is given by

$$\int_\Sigma \mathrm{d}\rho_0(e, m)\, f(e, m) = \int_{\mathbb{R}^6} \mathrm{d}e\ \mathrm{d}m\, \delta((e \cdot m)^2 - 1)\delta(|e|^2 - |m|^2) f(e, m). \tag{63}$$

4.2. Quasi-free states

Let $T(e, m)$ be a positive matrix, continuously depending on (e, m), large enough so that $T(e, m) + (\mathrm{i}/2)(e \cdot m)\epsilon(e, m)$ is a positive definite matrix (an appropriate 'covariant' choice of T will be discussed later on). Let w be a strictly positive function with integral 1 and let

$$\mathrm{d}\rho(e, m) = w(e, m)\, \mathrm{d}\rho_0(e, m). \tag{64}$$

The probability measure ρ together with the choice of T determines a state ω of $(\mathcal{A}, X, \sigma)$ by

$$\omega_{k,q;k',q'}(f) = \int_\Sigma \mathrm{d}\rho(e,m)\, f(e,m)\, \exp\left(\frac{1}{2}\mathrm{i}(e\cdot m)(k+\eta q)\cdot\epsilon(k'+\eta q')\right) \times \exp\left(-\frac{1}{2}(k-k'+\eta(q-q'))\cdot T(k-k'+\eta(q-q'))\right) \quad (65)$$

for all $k, k', q, q' \in \mathbb{R}^4$. The multiplier ξ associated with ω is given by

$$\xi(k,q;k',q') = \exp\left(\frac{1}{2}\mathrm{i}(e\cdot m)(k+\eta q)\cdot\epsilon(k'+\eta q')\right). \quad (66)$$

Note that in (65) k and k' have the interpretation of a shift in wave vector; q, q' an interpretation of a shift in position of the particle.

In the C^*-algebraic approach to quantum mechanics the notion of quasi-free states is introduced—see e.g. [13]. We have not yet studied its generalization to the present context of covariance systems. However, it is obvious that (65) is a good example of what can be a quasi-free state.

4.3. Position and momentum operators

Let $(\mathcal{H}, \pi, U, \psi)$ be the GNS-representation of $(\mathcal{A}, X, \sigma)$ induced by ω. From the continuity of (65) in k, k', q, q' it follows, using Stone's theorem, that U has self-adjoint generators. Because $\xi(k,q;k',q')$ commutes with $U(k'',q'')$ one can write

$$U(k,q) = \exp(-\mathrm{i}k\cdot\gamma Q + \mathrm{i}\gamma q\cdot K). \quad (67)$$

In the context of the GNS-construction, the operators Q_μ and K_μ can be written formally as a sum of a multiplication operator and a differential operator acting on functions $\psi(k,q)$ with a value in $\mathcal{C}_0(\Sigma)$. From the definition (see the proof of the GNS-construction) it follows that

$$U(k,q)\psi(k',q') = \psi(k+k', q+q')\xi(k',q';k,q). \quad (68)$$

Comparing the expansions of (67) and (68) one obtains

$$Q_\mu = \mathrm{i}\sum_\nu \gamma^{-1}_{\mu,\nu}\frac{\partial}{\partial k_\nu} + \frac{1}{2}q_\mu + \frac{1}{2}(\eta^{-1}k)_\mu \quad (69)$$

and

$$K_\mu = -\mathrm{i}\sum_\nu \gamma^{-1}_{\nu,\mu}\frac{\partial}{\partial q_\nu} + \frac{1}{2}k_\mu + \frac{1}{2}(\eta q)_\mu. \quad (70)$$

Using these explicit expressions it is easy to calculate the following commutation relations:

$$\begin{aligned}
[Q_\mu, Q_\nu] &= -\mathrm{i}(e\cdot m)(\gamma^{-1}\epsilon\tilde{\gamma}^{-1})_{\mu,\nu} \\
[K_\mu, K_\nu] &= \mathrm{i}(e\cdot m)\epsilon^{-1}_{\mu,\nu} \\
[K_\mu, Q_\nu] &= -\mathrm{i}\gamma^{-1}_{\nu,\mu}.
\end{aligned} \quad (71)$$

Note that the r.h.s. of these expressions is always an operator commuting with all the operators Q_μ and K_μ. Commutation relations of this kind have been proposed by Doplicher *et al* [14, 15] as a simplified model for quantum spacetime.

It is now very easy to calculate physically relevant quantities in the state ω, for example, one finds $(Q_\mu\psi, \psi) = 0$ and

$$(Q_\nu Q_\mu\psi, \psi) = \int_\Sigma \mathrm{d}\rho(e,m)\left[\gamma^{-1}\left(T(e,m) + \frac{1}{2}\mathrm{i}(e\cdot m)\epsilon(e,m)\right)\gamma^{-1}\right]_{\mu,\nu} \quad (72)$$

i.e. the state ω describes a particle with expected position at the origin of spacetime and with variance in position determined by $T \pm (\mathrm{i}/2)\epsilon$.

4.4. Lorentz transformations

In what follows we assume that

$$T(e', m') = \tilde{\Lambda} T(e, m)\Lambda \tag{73}$$

with (e', m') related to (e, m) by (62), i.e. T transforms like ϵ under the Lorentz transformations. Such an assumption is possible. Indeed, let C be any positive matrix. Fix a special point (e_0, m_0) in Σ, e.g. the point with $e = m = (1, 0, 0)$. Use it to define the matrix ϵ_0 by $\epsilon_0 = \epsilon(e_0, m_0)$. Now assume that $\tilde{\Lambda}\epsilon_0\Lambda = \epsilon_0$ implies that $\tilde{\Lambda} C\Lambda = C$. Then $T(e, m)$ is defined by

$$T(e, m) = \tilde{\Lambda} C\Lambda \qquad \text{whenever} \qquad \epsilon(e, m) = \tilde{\Lambda}\epsilon_0\Lambda. \tag{74}$$

Obviously, T transforms in the same way as ϵ. From

$$k \cdot T(e, m)k' = \Lambda k \cdot C\Lambda k' \tag{75}$$

it follows that $T(e, m)$ is positive. In fact, we need $T(e, m) + (\mathrm{i}/2)(e \cdot m)\epsilon(e, m) \geqslant 0$. This is satisfied if $C \pm (\mathrm{i}/2)\epsilon_0 \geqslant 0$; e.g. $C = \frac{1}{2}\mathbf{I}$ is a good choice.

If (73) is satisfied then a unitary representation R of the proper Lorentz group is defined by

$$R(\Lambda)\psi(k, q)(e, m) = \sqrt{\frac{w(e', m')}{w(e, m)}}\psi(\Lambda^{-1}k, \Lambda^{-1}q)(e', m') \tag{76}$$

with (e', m') as in (62). One calculates that

$$R(\Lambda)^*\psi(k, q)(e', m') = \sqrt{\frac{w(e, m)}{w(e', m')}}\psi(\Lambda k, \Lambda q)(e, m). \tag{77}$$

It is now straightforward to verify that $R(\Lambda)$ is unitary. One verifies that

$$\pi(\sigma_\Lambda f) = R(\Lambda)\pi(f)R(\Lambda)^*. \tag{78}$$

With the assumption that

$$\gamma(e', m') = \tilde{\Lambda}\gamma(e, m)\Lambda \tag{79}$$

i.e. γ also transforms in the same way as ϵ, one obtains

$$U(\Lambda k, \Lambda q) = R(\Lambda)U(k, q)R(\Lambda)^*. \tag{80}$$

Extension of the symmetry group X to include the proper Lorentz group is straightforward; e.g. let

$$\omega_{k,q,\Lambda;k',q',\Lambda'}(f) = (\pi(f)U(k, q)^*R(\Lambda)^*\psi, U(k', q')^*R(\Lambda')^*\psi). \tag{81}$$

5. Discussion

We have introduced the notion of a (mathematical) state of a covariance system $(\mathcal{A}, X, \sigma)$. It generalizes the notion of a covariant state of the C^*-algebra $\mathcal{A}$ by allowing that in the Hilbert space representation the action σ of the group X is implemented as a projective representation and by allowing that the cocycle associated with this representation is operator-valued (i.e. it is a C^*-multiplier). For these generalized covariant states we prove a GNS-theorem. It can be used as an alternative for working with crossed products of $\mathcal{A}$ and X. In fact, the proof captures all the essential elements of the proof of the existence of the crossed product algebra. The shift in emphasis from crossed product algebra to states of a covariance system has many advantages. In particular, the problem that the crossed product algebra depends on the C^*-multiplier is circumvented; e.g. spin of a quantum particle is a property of the states of a covariance system involving different C^*-multipliers. In the more traditional approach, the particles with or without spin are described by the states on different C^*-algebras.

A limitation of the present paper is that the representations involving anti-unitary operators are not included. Such representations are essential for physical applications. Wigner [1] showed that all the symmetry elements appearing in quantum mechanics must be implemented either by unitary or anti-unitary operators of the Hilbert space of wavefunctions. These anti-unitary operators appear only in the case of discrete symmetries. Indeed, one can always assume that the neutral element of the symmetry group is implemented as the identity operator. Then any element which is continuously connected with the neutral element must also be implemented as a unitary operator. We think that the omission of anti-unitary representations can be handled easily in most situations. It is clear from the examples that the physically relevant representation of the covariance system is usually already fixed by considering a subgroup of symmetries (e.g. the standard representation of quantum mechanics is already obtained by considering the subgroup of spatial shifts only). In such cases, the anti-unitary implementation of discrete symmetry elements can be added 'by hand', i.e. without relying on the GNS-theorem. A good example [4] is time-reversal symmetry, denoted by θ. When added to the Galilei group it anti-commutes with velocity v and time t and commutes with position q and rotation Λ. It is implemented as the anti-unitary operator which maps each wavefunction onto its complex conjugate. In the relativistic example, $\theta = -g$ is an element of the full Poincaré group. Its implementation is

$$R(\theta)\psi(k,q)(e,m) = \sqrt{\frac{w(-e,m)}{w(e,m)}}\,\overline{\psi(gk,\theta\bar{q})(-e,m)}. \tag{82}$$

The section on non-relativistic quantum mechanics picks up old ideas about the role of covariance systems in quantum mechanics and illustrates our approach to quantum mechanics and quantum field theory. It does not involve a quantization step referring to classical mechanics. It starts from a C^*-algebra $\mathcal{A}$ of 'classical' functions which are accessible for experimental measurement. A group X of symmetries, acting as automorphisms of $\mathcal{A}$, is considered. Together they form a covariance system; e.g. if $\mathcal{A}$ is a C^*-algebra of functions of position in $\mathbb{R}^3$ and X is the Euclidean group, then the standard representation of quantum mechanics is a representation of the covariance system $(\mathcal{A}, X, \sigma)$.

The section about the model of quantum spacetime has been added to give an example of projective representations involving operator-valued cocycles, i.e. non-trivial C^*-multipliers. The immediate effects of these are non-canonical commutation relations. Another aspect, well illustrated by this example, is how the GNS-construction can be used to build physically interesting representations starting from an explicit expression for a state of the covariance system. Also remarkable is the fact that the C^*-algebra $\mathcal{A}$ of 'classical' functions does not contain functions of position or momenta, but only functions of the internal degrees of freedom e and m. Hence, the properties of quantum spacetime appear only in the representation of the covariance system, the representation which depends on the state of the system. In this way we hope that the representations of many-particle systems may produce a non-flat spacetime structure. It is quite clear that our treatment of this model is far from complete. A full study is beyond the scope of the present paper and will be reported elsewhere [19].

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References

[1] Wigner E P 1931 *Spectra Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (Braunschweig)
(Wigner E P 1959 *Group Theory and its Applications to the Quantum Mechanics of Atomic Spectra* (New York: Academic))
[2] Moyal J E 1949 Quantum mechanics as a statistical theory *Proc. Camb. Phil. Soc.* **45** 99
[3] Bargmann V 1954 On unitary representations of continuous groups *Ann. Math.* **59** 1
[4] Schwinger J 1951 The theory of quantized fields I *Phys. Rev.* **82** 914
[5] Levy-Leblond J-M 1963 Galilei group and nonrelativistic quantum mechanics *J. Math. Phys.* **4** 776
[6] Doplicher S, Kastler D and Robinson D W 1966 Covariance algebras in field theory and statistical mechanics *Commun. Math. Phys.* **3** 1
[7] Busby R C and Smith H A 1970 Representations of twisted group algebras *Trans. Am. Math. Soc.* **149** 503
[8] Hagen C R 1970 The Bargmann–Wigner method in Galilean relativity *Commun. Math. Phys.* **18** 97
[9] Hagen C R 1971 Galilean invariant Lee model for all spins and parities *Commun. Math. Phys.* **21** 219
[10] Takesaki M 1979 *Theory of Operator Algebras* vol I (Berlin: Springer) chapter V, section 7
[11] Bratteli O and Robinson D W 1997 *Operator Algebras and Quantum Statistical Mechanics* vol I (New York: Springer)
[12] Landsman N P 1990 Quantization and superselection sectors I. Transformation group C^*-algebras *Rev. Math. Phys.* **2** 45
Landsman N P 1990 Quantization and superselection sectors II. Dirac monopole and Aharonov–Bohm effect *Rev. Math. Phys.* **2** 73
[13] Petz D 1990 *An Invitation to the Algebra of Canonical Commutation Relations* (Leuven: Leuven University Press)
[14] Doplicher S, Fredenhagen K and Roberts J E 1994 Spacetime quantization induced by classical gravity *Phys. Lett.* B **331** 39
[15] Doplicher S, Fredenhagen K and Roberts J E 1995 The quantum structure of spacetime at the Planck scale and quantum fields *Commun. Math. Phys.* **172** 187
[16] Rieffel M A 1997 On the operator algebra for the space-time uncertainty relations *Operator Algebras and Quantum Field Theory* ed S Doplicherd, R Longo, J E Roberts and L Zsido (Providence, RI: American Mathematical Society) pp 374–82
[17] Landsman N P 1998 *Mathematical Topics Between Classical and Quantum Mechanics* (New York: Springer)
[18] Naudts J 1999 C^*-Multipliers, crossed product algebras, and canonical commutation relations *Preprint* math-ph/9907008
[19] Naudts J and Kuna M 2001 *J. Phys. A: Math. Gen.* **34** 4227
(Naudts J and Kuna M 2000 *Preprint* hep-th/0012209)